On Wick algebras with additional twisted commutation relations

1997 J. Phys. A: Math. Gen. 30 3235

(http://iopscience.iop.org/0305-4470/30/9/029)

J. Phys. A: Math. Gen. **30** (1997) 3235–3247. Printed in the UK PII: S0305-4470(97)78138-1

On Wick algebras with additional twisted commutation relations

Robert Rałowski†

Institute of Theoretical Physics, University of Wrocław, Pl. Maxa Borna 9, 50-204 Wrocław, Poland

Received 23 December 1996

Abstract. We study the construction of the C-symmetric creation and annihilation operators, which form a Wick algebra with some additional relations. This Wick algebra is an algebra of the operators which act on the associative algebra of polynomials with noncommutative variables $F[x_1, \ldots, x_n]$ divided by an ideal generated by quadratic relations between the variables $x_1, \ldots, x_n$. As a result the consistency conditions for such a representation are obtained. The creators and annihilator have to preserve ideal $I = \langle x_i x_j - \sum b_{i,j}^{k,l} x_k x_l \rangle$, where $b_{i,j}^{k,l} \in \mathcal{C}$, which leads us to consistency conditions.

1. Introduction

We know that all standard particles in physics can be divided into two statistics, bosons and fermions, but many new statistics of the particle excitations have recently been discovered in two-dimensional systems of the particles. These new statistics have the following commutation relations for annihilation and creation operators:

$$a_i a_j^+ - q a_j^+ a_i = \delta_{i,j} \mathbf{1}$$

here $-1 \leqslant q \leqslant 1$ as interpolation between bosons and fermions was studied by Greenberg [1, 2], Mohopatra [3], and Bożejko and Speicher [4]. Above q we do not have additional relation commutations between creators (then annihilators), but if we take $q \in \mathcal{C}$ such that $k \in \mathcal{N}$, $q^{k(k+1)} = 1$ exist, then we have additional relations (see [4]). If the deformation parameter, q, is a root of the unity, then such statistics have an amiable physical interpretation, as studied by Wilczek [21, 22] and Wu [23]. In this case a particle equipped with charge e is moving on the plane around a singular magnetic field which is perpendicular to our plane. Every rotation gives a phase factor in the wavefunction of the particle. Some authors called these statistics anyones. A certain generalization is given by Marcinek (see [28]) as quon statistic, see example 9. In [28] Marcinek considered a system of N particles moving around N magnetic singularities.

We would like to present the construction of deformed algebras of commutation relations. Examples of deformed commutation relation, i.e. interpolations between bosonic and fermionic statistics, were studied in [4, 5, 1, 6, 8]. The problem of additional relations between pairs of annihilation operators and between creation operators was considered, for example, in [9, 10]. In this paper we construct a Wick algebra with additional commutation relations, and we prove consistency conditions for the existence of such

† E-mail address: rober@ift.uni.wroc.pl

algebras. The construction of annihilation operators is analogous to partial differentation in the noncommutative differential calculus [11, 14, 15] applied to quantum groups, see [16, 18–20]. But construction of these operators is not exactly the same as our construction. The representation of the $SU_q(n)$ algebra with twisted commutation relations can be obtained from the q-oscillators Fock representation (which satisfies our consistency conditions). This construction is given by Chaichian *et al* (see [13, 12]). The $SU_q(n)$ representation was studied by Chaichian *et al* (see [17]). Another deformation of the boson statistic is described by Farlie and Nuyts (see [7]) as a q derivation, see example 7 in section 5.

Generally, in Wick algebra there are no relations between creation (or annihilation) operators themselves but such relations are possible in certain cases. Obviously all such relations should be consistent. Hence, we need some additional assumption, see [9]. In this paper we are going to study an explicit construction of commutation relations and consistency conditions. Our construction of commutation relations is based on the operators C, B and $\tilde{B}$. These operators are not arbitrary, they must satisfy some consistency conditions such as Wess–Zumino conditions for differential calculus on a quantum plane [15]. As a result we obtain the consistency conditions from the assumption that the equivalence class of the creation and annihilation operators of the relation defined by the equation $x - Bx = 0$, $x \in F_2[x_1, \ldots, x_n]$ must be constant.

In section 2 we apply the notion of contraction (called evaluation in [24, 25]) defined in an algebraic way, see [29], on the polynomials algebra $F[x_1, \ldots, x_n]$ of the noncommutative variables $x_1, \ldots, x_n$ by taking a generalized twist, C, on $F_2[x_1, \ldots, x_n]$. This contraction satisfies C-deformed Leibniz rule. In section 2 we introduce a partial representation of the creation and annihilation operators defined on the algebra $F[x_1, \ldots, x_n]$. In this representation we have commutation relations between creators and annihilators. In this way we obtain a Wick algebra in which any sequences consisting of creation and annihilation operators can be arranged in Wick's way, see [9]. If we want to obtain additional relations, we have to divide the polynomial algebra $F[x_1, \ldots, x_n]$ by an $\mathcal{N}$-graded ideal $I \subset F[x_1, \ldots, x_n]$, generated by the twist $B \in \mathrm{End}(F_2[x_1, \ldots, x_n])$ and construct the subspace $J_2^* \subset F_2[x_1, \ldots, x_n]$, generated by the twist $\tilde{B} \in \mathrm{End}(F_2[x_1, \ldots, x_n])$. Then we construct a representation of the annihilation and creation on the algebra $\mathcal{A} = F[x_1, \ldots, x_n]/I$, which is projected from the representation defined on the algebra $F[x_1, \ldots, x_n]$. If we want to make the above construction from the twists B, C, they have to satisfy certain relations, from which it appears that the partial representation must preserve the ideal, I. In this way we obtain a noncommutative differential calculus on the quantum plane, see [15]. The crucial point to prove is the additional commutation relation between annihilators is the π^*—invariant property for contraction introduced in section 2. In section 3 we present deformed commutation relations:

$$[d_i, d_j^+]_C = \delta_{i,j}\mathbf{1} \qquad [d_i^+, d_j^+]_B = 0 \qquad [d_i, d_j]_{\tilde{B}} = 0$$

as a consequence of definitions introduced in section 2.

To obtain a representation on $\mathcal{A}$, the operators defined on the free algebra $F[x_1, \ldots, x_n]$ have to preserve the ideal contained in $F[x_1, \ldots, x_n]$, which leads us to the consistency conditions between the tensors $B, \tilde{B}, C$. The conditions which satisfy the assumptions of theorem 3.3 are enough to construct the representation defined on the algebra $\mathcal{A}$. In this case the tensors B, C satisfy generalized braided symmetry constructed in [11, 14], where there are similar consistency conditions. The contraction has the π^*-invariant property when tensors $\tilde{B}, C$ satisfy the assumptions of theorem 3.4 and in the particular case see corollary 3.1.

In section 5 we present an example which satisfies the assumptions of the theorems in section 3. Of course, bosons and fermions satisfy these assumption, however, we have shown a q-deformed example. Let us observe that the colour statistics form a Wick algebra with an additional relation on the Fock space in which creators are adjoints to the annihilators.

2. Definitions and notions

Let $F[x_1, \ldots, x_n]$ be an associative free algebra of polynomials with noncommutative variables over the field $\mathcal{C}$. The monomial $x_{i(1)}^{n_1} \ldots x_{i(k)}^{n_k}$ is of mth degree if $\sum_{i=1}^{k} n_i = m$ for $1 \leqslant i(l) \leqslant n$ for $l \in \{1, \ldots, k\}$ and $i(l) \neq i(l+1)$. Let $F_m[x_1 \ldots x_n] = F_m$ be the set of linear combinations of monomials at mth degree. It is easy to see that the algebra of polynomials is an $\mathcal{N}$-graded algebra:

$$F[x_1 \ldots x_n] = \bigoplus_{m \geqslant 0} F_m \qquad \text{where } F_0[x_1 \ldots x_n] = F_0 = \mathcal{C}.$$

Now let I be a two-sided minimal ideal generated by the elements:

$$x_i x_j - \sum_{k=1,l=1}^{n} b_{i,j}^{k,l} x_k x_l \qquad \text{where } b_{i,j}^{k,l} \in \mathcal{C} \text{ for } i, j, k, l \in \{1, \ldots, n\}.$$

This ideal has the following form: $I = \bigoplus_{m \geqslant 0} I_m$, where $I_{m+1} = F_1[x_1, \ldots, x_n] I_m + I_m F_1[x_1, \ldots, x_n]$ for $2 \leqslant m$ and $I_2 = \operatorname{Im}(\mathbf{1} - B) \subset F_2[x_1, \ldots, x_n]$, $I_1 = \{0\} \in F_1[x_1, \ldots, x_n]$ and $I_0 = \{0\} \in \mathcal{C}$.

Let us define the quotient algebra of n variables $x_1, \ldots, x_n$. $\mathcal{A}[x_1, \ldots, x_n] := F[x_1, \ldots, x_n]/I$. From the above considerations, it follows that $\mathcal{A} = \bigoplus_{m \geqslant 0} F_m[x_1, \ldots, x_n] / I_m = \bigoplus_{m \geqslant 0} \mathcal{A}_m$.

We will introduce a few important definitions for our paper.

Definition 2.1. A triple $(\mathcal{W}, \boldsymbol{k}, C)$ is said to be Wick algebra iff $(\mathcal{W}, \boldsymbol{k})$ is an algebra with unity $\mathbf{1}$ over field $\boldsymbol{k}$, generated by $x_1, \ldots x_n, x_1^+ \ldots x_n^+$ with relations:

$$x_i x_j^+ - \sum_{k,l} c_{i,j}^{k,l} x_k^+ x_l = \delta_{i,j} \mathbf{1} \qquad \boldsymbol{k} \ni c_{i,j}^{k,l} \neq 0 \text{ for finitely many } k, l.$$

Definition 2.2. A linear map $C \in \operatorname{End}(F_2[x_1, \ldots, x_n])$ is said to be a twist with matrix elements denoted by $c_{i,j}^{k,l} \in \mathcal{C}$, where $i, j, k, l \in \{1, \ldots, n\}$.

The above definition allows us to define a map $C_m^{(i)} \in \operatorname{End}(F_m[x_1, \ldots, x_n])$, by the formulae:

$$C_m^{(i)}(x_{j(1)} \ldots x_{j(i)} x_{j(i+1)} \ldots x_{j(m)}) = x_{j(1)} \ldots C(x_{j(i)} x_{j(i+1)}) \ldots x_{j(m)}$$

for $i \in \{1, \ldots, m-1\}$ and let $C_m^{(0)} = \mathrm{id} = \mathbf{1}$ be identity on $F_m[x_1, \ldots, x_n]$.

Definition 2.3. A linear mapping $g \in \operatorname{Hom}(F_2[x_1, \ldots, x_n], \mathcal{C})$ is said to be pairing iff we have the following relation

$$F_2[x_1, \ldots, x_n] \ni x_i x_j \longmapsto g(x_i x_j) = \delta_{i,j} \in \mathcal{C} \qquad \text{for } i, j \in \{1, \ldots, n\}.$$

Definition 2.4. The C-twisted contraction h_m with respect to elementary twist C is a mapping $h_m \in \operatorname{Hom}(F_{m+1}[x_1, \ldots, x_n], F_{m-1}[x_1, \ldots, x_n])$ such that:

$$h_m = \sum_{k=1}^{m} (\mathbf{1}_{n-1} \otimes g \otimes \mathbf{1}_{m-k}) C_m^{(k-1)} \ldots C_m^{(0)}.$$

Let us assume that

$$h_M^{(k)} := \mathbf{1}_{k-1} \otimes h_m \in \mathrm{End}(F_{m+k}[x_1, \ldots, x_n]_1 F_{m+k-2}[x_1, \ldots, x_n]).$$

Directly from the definition of the contraction we have the following proposition.

Proposition 2.1. The contraction h_n has the following properties:

(1) C-Leibniz rule:

$$h_n^{(k)} = h_1^{(k)} + h_{n-1}^{(k+1)} C_n^{(k)}$$

(2) $h_n^{(1)} = h_m^{(1)} + h_{n-m}^{(m+1)} C_n^{(m)} \ldots C_n^{(1)}$.

Proof. We will only prove $k = 1$ by induction on n. For $n = 1$ the C-Leibniz rule is obvious. Then let $y \in F_{n-1}[x_1, \ldots, x_n]$ and $x_i, x_j \in F_1[x_1, \ldots, x_n]$, then we have:

$$\begin{aligned}
h_{n+1}^{(1)}(x_i x_j y) &= h_1^{(1)}(x_i x_j y) + \sum_{s=2}^{n+1} h_1^{(s)} C^{(s-1)} \ldots C^{(1)}(x_i x_j y) \\
&= h_1^{(1)}(x_i x_j y) + \sum_{s=2}^{n+1} \sum_{k,l=1}^{n} c_{i,j}^{k,l} h_1^{(s)} C^{(s-1)} \ldots C^{(2)}(x_k x_l y) \\
&= h_1^{(1)}(x_i x_j y) + \sum_{k,l=1}^{n} c_{i,j}^{k,l} x_k \sum_{s=1}^{n} h_1^{(s)} C^{(s-1)} \ldots C^{(1)}(x_l y) \\
&= h_1^{(1)}(x_i x_j y) + \sum_{k,l=1}^{n} c_{i,j}^{k,l} x_k h_n^{(1)}(x_l y) \\
&= h_1^{(1)}(x_i x_j y) + \sum_{k,l=1}^{n} c_{i,j}^{k,l} h_{n+1}^{(2)}(x_k x_l y) \\
&= h_1^{(1)}(x_i x_j y) + h_{n+1}^{(2)} C^{(1)}(x_i x_j y).
\end{aligned}$$

The proof of the next formula goes similarly. □

Now we will introduce C-partial creation and annihilation operators.

Definition 2.5. Operators $a_i, a_i^+ \in \mathrm{End}(F[x_1, \ldots, x_n])$ defined by formulae

$$a_{m,i}(y) \stackrel{\text{def}}{=} h_m(x_i y) \qquad a_{m,i}^+(y) \stackrel{\text{def}}{=} x_i y$$

for $y \in F_m[x_1, \ldots, x_n]$ and $i \in \{1, \ldots, n\}$ are said to be annihilation and creation ones.

Let us define:

$$J_2^* = \mathrm{Im}(\mathbf{1} - \tilde{B}) \subset F_2[x_1, \ldots, x_n] \qquad \text{where } \tilde{B} \in \mathrm{End}(F_2[x_1, \ldots, x_n]).$$

The crucial notion in our paper is a mapping defined on $F[x_1, \ldots, x_n]$.

Definition 2.6. Let $\omega \in \mathrm{End}(F[x_1, \ldots, x_n])$ be a mapping such that

$$F_m[x_1, \ldots, x_n] \ni y \longmapsto \omega(y) \in F_{m-1}[x_1, \ldots, x_n] \qquad \text{for } m \in \mathcal{N} \text{ and } \omega(\mathcal{C}) = 0.$$

Then we will say that ω is π^*-invariant on $F_m[x_1, \ldots, x_n]$ if:

$$\omega \circ \omega(J_2^* F_m[x_1, \ldots, x_n]) \subset I_{m-2}.$$

Definition 2.7. Let $d_i, d_i^+ \in \mathrm{End}(F[x_1, \ldots, x_n])$, then $\{d_i\}_{i=1}^n, \{d_i^+\}_{i=1}^n$ is a π^*-invariant representation of annihilation and creation operators iff the C-partial operators $\{a_i\}_{i=1}^n$, $\{a_i^+\}_{i=1}^n$, exist which satisfy the following conditions.

(1) The following diagrams commute

$$\begin{array}{ccc} F_m[x_1,\ldots,x_n] & \xmapsto{a_{m,i}} & F_{m-1}[x_1,\ldots,x_n] \\ \pi_m \big\downarrow & & \big\downarrow \pi_{m-1} \\ \mathcal{A}_m & \xmapsto[d_{m,i}]{} & \mathcal{A}_{m-1} \end{array} \qquad \begin{array}{ccc} F_m[x_1,\ldots,x_n] & \xmapsto{a^+_{m,i}} & F_{m+1}[x_1,\ldots,x_n] \\ \pi_m \big\downarrow & & \big\downarrow \pi_{m+1} \\ \mathcal{A}_m & \xmapsto[d^+_{m,i}]{} & \mathcal{A}_{m+1} \end{array}.$$

(2) The operators a_i are π^*-invariant operators.

3. Commutation relations

In this section we would like to calculate commutators that are introduced in the following way:

$$[t_i, t_j]_D \stackrel{\text{def}}{=} t_i t_j - \sum_{k,l=1}^{n} d_{i,j}^{k,l} t_k t_l$$

for $t_i \in \mathrm{End}(F[x_1, \ldots, x_n])$ and $d_{i,j}^{k,l} \in \mathcal{C} i, j, k, l \in \{1, \ldots, n\}$.

We then have the following theorem.

Theorem 3.1. Let $\{d_i\}_{i=1}^n, \{d_i^+\}_{i=1}^n$ be a π^*-invariant representation of creation and annihilation operators which are acting on the quotient algebra $\mathcal{A}$, then we have the following commutation relations:

$$[d_i, d_j^+]_C = \delta_{i,j}\mathbf{1} \qquad [d_i, d_j]_{\tilde{B}} = 0 \qquad [d_i^+, d_j^+]_B = 0.$$

Proof. Now we will prove the first relation. Let $y \in F_n[x_1, \ldots, x_n]$, $n \in N$. Then by applying the C-Leibniz rule we obtain:

$$\begin{aligned} a_{n+1,i} \circ a_{n,j}^+(y) &= a_{n+1,i}(x_j y) \\ &= h_{n+1}^{(1)}(x_i x_j y) \\ &= h_1^{(1)}(x_i x_j y) + h_n^{(2)} \circ C^{(1)}(x_i x_j y) \\ &= \delta_{i,j} y + \sum_{k,l} c_{i,j}^{k,l}(x_k(h_n(x_l y))) \\ &= [\delta_{i,j} + \sum_{k,l} c_{i,j}^{k,l}(a_{n-1,k}^+ \circ a_{n,l})](y) \end{aligned}$$

and for $y_1 \in \mathcal{A}_n$ we have by the commutative diagrams the following equalities

$$\begin{aligned} d_{n+1,i} \circ d_{n,j}^+ y_1 &= d_{n+1,i} \circ d_{n,j}^+ \pi_n y \\ &= \pi_n a_{n+1,i} \circ a_{n,j}^+ y \\ &= \sum_{k,l} c_{i,j}^{k,l} \pi_n a_{n-1,k}^+ \circ a_{n,l} y + \delta_{i,j} \pi_n y \\ &= \sum_{k,l} c_{i,j}^{k,l} d_{n-1,k}^+ \circ d_{n,l} y_1 + \delta_{i,j} y_1. \end{aligned}$$

For the second one let $y_1 \in A_n$ and $y_1 = \pi_n(y)$ for $y \in F_n[x_1, \dots, x_n]$. By using definition 2.7 and π^*-invariance it follows:

$$\begin{aligned} d_{n-1,i} \circ d_{n,j}(y_1) &= \pi_{n-2} \circ a_{n-1,i} \circ a_{n,j}(y) \\ &= \pi_{n-2} h^{(1)}_{n-1} h^{(2)}_n (x_i x_j y) \\ &= \pi_{n-2} h^{(1)}_{n-1} h^{(2)}_n (x_i x_j - \tilde{B}(x_i x_j)) y + \pi_{n-2} h^{(1)}_{n-1} h^{(2)}_n \tilde{B}(x_i x_j) y \\ &= \pi_{n-2} z + \sum_{k,l} \tilde{b}^{k,l}_{i,j} \pi_{n-2} h^{(1)}_{n-1} h^{(2)}_n (x_k x_l y) \\ &= \sum_{k,l} \tilde{b}^{k,l}_{i,j} d_{n-1,k} \circ d_{n,l}(y_1) \end{aligned}$$

where $z \in I_{n-2}$. The last one follows from the commutation relation of the quotient algebra, which completes the proof. □

The operators d_i, d^+_j form a Wick algebra. But there is a problem with the existence of such operators which have additional commutation relations. From the definition of representation d_i, d^+_j, for $i, j \in \{1, \dots, n\}$ (see definition 2.7) it follows that there exists a C-partial representation which satisfies the commutative diagrams (from the previous section). Thus, it follows that C-partial representation have to preserve a two-sided ideal, I, generated by quadratic relations $x_i x_j - \sum_{k,l=1}^n b^{k,l}_{i,j} x_k x_n = 0$.

From this we have neccessary conditions.

Proposition 3.2. A necessary condition for the constant action on the equivalence classes of the operator a_i is q

$$(\mathbf{1} - B)(\mathbf{1} + \tilde{C}) = 0$$

where $\tilde{C}$ has the matrix elements given by $\tilde{c}^{k,l}_{i,j} = c^{i,k}_{j,l}$.

Proof. By taking variables x_j, x_k of the space $F_1[x_1, \dots, x_n]$ and by using the C-Leibniz rule (see lemma 2.1) we have:

$$\begin{aligned} \bar{0} &= h_2(x_i(\mathbf{1} - B)(x_j x_k)) \\ &= \sum_{l,m} (\delta_{j,l}\delta_{k,m} - b^{l,m}_{j,k}) h_2(x_i x_l x_m) \\ &= \sum_{l,m} (\delta_{j,l}\delta_{k,m} - b^{l,m}_{j,k}) (h^{(1)}_1 + h^{(2)}_1 C^{(1)})(x_i x_l x_m) \\ &= \sum_{l,m} (\delta_{j,l}\delta_{k,m} - b^{l,m}_{j,k}) \Big(\delta_{i,l} x_m + \sum_{r,s} c^{r,s}_{i,l} h^{(2)}_1 (x_r x_s x_m)\Big) \\ &= \sum_{l,m} (\delta_{j,l}\delta_{k,m} - b_{j,k,l,m}) \sum_r (\delta_{i,l}\delta_{m,r} x_r + c^{r,m}_{i,l} x_r) \\ &= \sum_r \sum_{l,m} (\delta_{j,l}\delta_{k,m} - b^{l,m}_{j,k})(\delta_{i,l}\delta_{m,r} + c^{r,m}_{i,l}) x_r. \end{aligned}$$

Then we have:

$$\sum_{l,m} (\delta_{j,l}\delta_{k,m} - b^{l,m}_{j,k})(\delta_{i,l}\delta_{m,r} + c^{r,m}_{i,l}) = 0.$$

□

The next theorem gives sufficient conditions for the existence of a C-representation which is acting on the quotient algebra $\mathcal{A}$ with relations: $[d_i, d^+_j]_C = \delta_{i,j}\mathbf{1}$ and $[d^+_i, d^+_j]_B = 0$.

Theorem 3.3 (Borowiec–Kharchenko). If the tensors $B, C \in \mathrm{End}(F_2[x_1, \ldots, x_n])$ satisfy the following conditions:

(1) $(\mathbf{1} - B)(\mathbf{1} + \tilde{C}) = 0$ where $\tilde{c}^{k,l}_{i,j} = c^{i,k}_{j,l}$

(2) there exists $A \in \mathrm{End}(F_3[x_1, \ldots, x_n])$ such that: $C^{(2)}C^{(1)}B^{(2)} - B^{(1)}C^{(2)}C^{(1)} = (\mathbf{1} - B^{(1)})A$

then $a_r(I) \subset I$ where $I \subset TE$ is the minimal ideal generated by the quadratic relation $x - Bx = 0$ for $x \in F_2[x_1, \ldots, x_n]$.

Proof of this theorem was given in [11, 26].

If we want to have commutation relations between annihilators, then the next theorem shows when it is possible.

Theorem 3.4. If matrices $\tilde{B}, C \in \mathrm{End}(F_2[x_1, \ldots, x_n])$ satisfy the following conditions:

(1) they fulfil YBE: $\tilde{B}^{(2)}C^{(1)}C^{(2)} = C^{(1)}C^{(2)}\tilde{B}^{(1)}$

(2) $[h^{(1)}_1 h^{(k)}_1 C^{(k-1)} \ldots C^{(2)} + h^{(k-2)}_1 C^{(k-3)} \ldots C^{(1)} h^{(2)}_1][\mathbf{1} - \tilde{B}^{(1)}] = 0$ for $k \in \{3 \ldots n\}$ and $n \in \mathcal{N}$

then $h^{(1)}_{m-1} h^{(2)}_m (J^*_2 F_m[x_1, \ldots, x_n]) \subset I_{m-2} \subset F_{m-2}[x_1, \ldots, x_n]$.

Proof. We will prove this by induction. Let us suppose that for certain $m - 1 \in \mathcal{N}$ the assertion is satisfied. Then

$$
\begin{aligned}
h^{(1)}_{m-1} h^{(2)}_m (\mathbf{1} - \tilde{B}^{(1)}) &= (h^{(1)}_1 + h^{(2)}_{m-2} C^{(1)})(h^{(2)}_1 + h^{(3)}_{m-1} C^{(2)})(\mathbf{1} - \tilde{B}^{(1)}) \\
&= (h^{(1)}_1 h^{(2)}_m + h^{(2)}_{m-2} C^{(1)} h^{(2)}_1)(\mathbf{1} - \tilde{B}^{(1)}) + h^{(2)}_{m-2} C^{(1)} h^{(3)}_{m-1} C^{(2)} (\mathbf{1} - \tilde{B}^{(1)}) \\
&= (h^{(1)}_1 h^{(2)}_m + h^{(2)}_{m-2} C^{(1)} h^{(2)}_1)(\mathbf{1} - \tilde{B}^{(1)}) + h^{(2)}_{m-2} h^{(3)}_{m-1} C^{(1)} C^{(2)} (\mathbf{1} - \tilde{B}^{(1)}) \\
&= (h^{(1)}_1 h^{(2)}_m + h^{(2)}_{m-2} C^{(1)} h^{(2)}_1)(\mathbf{1} - \tilde{B}^{(1)}) + h^{(2)}_{m-2} h^{(3)}_{m-1} (\mathbf{1} - \tilde{B}^{(2)}) C^{(1)} C^{(2)}.
\end{aligned}
$$

The last term belongs to I_{m-2}. Hence

$$
\begin{aligned}
h^{(1)}_{m-1} h^{(2)}_m (\mathbf{1} - \tilde{B}^{(1)}) &= (h^{(1)}_1 h^{(2)}_m + h^{(2)}_{m-2} C^{(1)} h^{(2)}_1)(\mathbf{1} - \tilde{B}^{(1)}) + z \\
&= (h^{(1)}_1 h^{(2)}_m + h^{(2)}_{m-2} C^{(1)} h^{(2)}_1)(\mathbf{1} - \tilde{B}^{(1)}) \\
&\quad + (h^{(1)}_1 h^{(2)}_1 - h^{(1)}_1 h^{(2)}_1)(\mathbf{1} - \tilde{B}^{(1)}) + z \\
&= (h^{(1)}_1 h^{(2)}_m + h^{(1)}_{m-1} h^{(2)}_1 - h^{(1)}_1 h^{(2)}_1)(\mathbf{1} - \tilde{B}^{(1)}) + z \\
&= h^{(1)}_1 (h^{(2)}_1 + h^{(3)}_1 C^{(2)} + \cdots + h^{(m+1)}_1 C^{(m)} \ldots C^{(2)}) \\
&\quad + (h^{(1)}_1 + h^{(2)}_1 C^{(1)} + \cdots + h^{(m-1)}_1 C^{(m-2)} \ldots C^{(1)}) h^{(2)}_1)(\mathbf{1} - \tilde{B}^{(1)}) \\
&\quad - h^{(1)}_1 h^{(2)}_1 (\mathbf{1} - \tilde{B}^{(1)}) + z
\end{aligned}
$$

where $z \in I_{m-2}$. We get the assertion from condition 2.

To finish the proof we only need to carry out the calculation for $m = 2$

$$
\begin{aligned}
h^{(1)}_1 h^{(2)}_2 (\mathbf{1} - \tilde{B}^{(1)}) &= h^{(1)}_1 (h^{(2)}_1 + h^{(3)}_1 C^{(2)})(\mathbf{1} - \tilde{B}^{(1)}) \\
&= (h^{(1)}_1 h^{(2)}_1 + h^{(1)}_1 h^{(3)}_1 C^{(2)})(\mathbf{1} - \tilde{B}^{(1)}) = 0.
\end{aligned}
$$

□

Let us consider a simpler situation, i.e. one in which the matrix has the following form $c^{k,l}_{i,j} = c_{i,j}\delta_{i,l}\delta_{j,k}$ and so on. Then we have a simple corollary on the existence of such a representation.

Corollary 3.1. Let $c^{k,l}_{i,j} = c_{i,j}\delta_{i,l}\delta_{j,k}$ and $\tilde{b}^{k,l}_{i,j} = \tilde{b}_{i,j}\delta_{i,l}\delta_{j,k}$ and $b^{k,l}_{i,j} = b_{i,j}\delta_{i,l}\delta_{j,k}$, then the tensors $B, \tilde{B}, C$ define a π^*-representation of creation and annihilation operators if the following conditions are satisfied:

(1) $(\mathbf{1}-B)(\mathbf{1}+\tilde{C})=0$ where $\tilde{c}_{i,j}^{k,l}=c_{j,l}^{i,k}$
(2) $c_{i,j}-\tilde{b}_{j,i}=0$ and $1-c_{i,j}\tilde{b}_{i,j}=0$ for $i,j\in\{1,\ldots,n\}$.

Proof. Because the tensors satisfy the Yang–Baxter equation, we have to prove that the second condition of theorem 3.4 is satisfied

$$\begin{aligned}(h_1^{(1)}h_1^{(k)}C^{k-1}\ldots C^2+h_1^{k-2}C^{k-3}\ldots C^1h_1^2)(\mathbf{1}-\tilde{B}^1)x_{i_1}x_{i_2}x_{j_1}\ldots x_{j_n}\\ &= h_1^{(1)}h_1^{(k)}x_{i_1}x_{j_1}\ldots x_{j_{k-2}}x_{i_2}x_{j_{k-1}}\ldots x_{j_n}c_{i_2,j_1}\ldots c_{i_2,j_{k-2}}\\ &\quad+\delta_{i_2,j_1}h_1^{(k-2)}x_{j_2}\ldots x_{j_{k-2}}x_{i_1}x_{j_{k-1}}\ldots x_{j_n}c_{i_1,j_2}\ldots c_{i_1,j_{k-2}}\\ &\quad-\tilde{b}_{i_1,i_2}[h_1^{(1)}h_1^{(k)}x_{i_2}x_{j_1}\ldots x_{j_{k-2}}x_{i_1}x_{j_{k-1}}\ldots x_{j_n}c_{i_1,j_1}\ldots c_{i_1,j_{k-2}}\\ &\quad+\delta_{i_1,j_1}h_1^{(k-2)}x_{j_2}\ldots x_{j_{k-2}}x_{i_2}x_{j_{k-1}}\ldots x_{j_n}c_{i_2,j_2}\ldots c_{i_2,j_{k-2}}]\\ &= [\delta_{i_1,j_1}\delta_{i_2,j_{k-1}}c_{i_2,j_1}\ldots c_{i_2,j_{k-2}}+\delta_{i_2,j_1}\delta_{i_1,j_{k-1}}c_{i_1,j_2}\ldots c_{i_1,j_{k-2}}\\ &\quad-\tilde{b}_{i_1,i_2}[\delta_{i_2,j_1}\delta_{i_1,j_{k-1}}c_{i_1,j_1}\ldots c_{i_1,j_{k-2}}\\ &\quad+\delta_{i_1,j_1}\delta_{i_2,j_{k-1}}c_{i_2,j_2}\ldots c_{i_2,j_{k-2}}]]x_{j_2}\ldots\hat{x}_{j_{k-1}}\ldots x_{j_n}\\ &= [\delta_{i_1,j_1}\delta_{i_2,j_{k-1}}(c_{i_2,j_1}-\tilde{b}_{i_1,i_2})c_{i_2,j_2}\ldots c_{i_2,j_{k-2}}\\ &\quad+\delta_{i_2,j_1}\delta_{i_1,j_{k-1}}(1-c_{i_1,j_1}\tilde{b}_{i_1,i_2})c_{i_1,j_2}\ldots c_{i_1,j_{k-2}}]x_{j_2}\ldots\hat{x}_{j_{k-1}}\ldots x_{j_n}\\ &= 0x_{j_2}\ldots\hat{x}_{j_{k-1}}\ldots x_{j_n}.\end{aligned}$$

□

By taking above formula in the proof of corollary 3.1:

$$\begin{aligned}(h_1^{(1)}h_1^{(k)}C^{k-1}\ldots C^2+h_1^{k-2}C^{k-3}\ldots C^1h_1^2)(\mathbf{1}-\tilde{B}^1)x_{i_1}x_{i_2}x_{j_1}\ldots x_{j_n}\\ &= [\delta_{i_1,j_1}\delta_{i_2,j_{k-1}}(c_{i_2,j_1}-\tilde{b}_{i_1,i_2})c_{i_2,j_2}\ldots c_{i_2,j_{k-2}}\\ &\quad+\delta_{i_2,j_1}\delta_{i_1,j_{k-1}}(1-c_{i_1,j_1}\tilde{b}_{i_1,i_2})c_{i_1,j_2}\ldots c_{i_1,j_{k-2}}]x_{j_2}\ldots\hat{x}_{j_{k-1}}\ldots x_{j_n}\end{aligned}$$

and assuming that $i_1=j_1=i_2$ we have a nonzero term only if $i_1=j_{k-1}$, then we have

$$\begin{aligned}(c_{i_1,i_1}-\tilde{b}_{i_1,i_1}+1-c_{i_1,i_1}\tilde{b}_{i_1,i_1})c_{i_2,j_2}\ldots c_{i_2,j_{k-2}}x_{j_2}\ldots\hat{x}_{j_{k-1}}\ldots x_{j_n}\\ &= (1+c_{i_1,i_1})(1-\tilde{b}_{i_1,i_1})c_{i_2,j_2}\ldots c_{i_2,j_{k-2}}x_{j_2}\ldots\hat{x}_{j_{k-1}}\ldots x_{j_n}.\end{aligned}$$

Then by putting $(1+c_{i,i})(1-\tilde{b}_{i,i})=0$ and for $i\neq j$ $c_{i,j}-\tilde{b}_{j,i}=0$ and $1-c_{i,j}\tilde{b}_{i,j}=0$ we have the following corollary.

Corollary 3.2. Let $c_{i,j}^{k,l}=c_{i,j}\delta_{i,l}\delta_{j,k}$ and $\tilde{b}_{i,j}^{k,l}=\tilde{b}_{i,j}\delta_{i,l}\delta_{j,k}$ and $b_{i,j}^{k,l}=b_{i,j}\delta_{i,l}\delta_{j,k}$, then the tensors $B,\tilde{B},C$ define a π^*-representation of creation and annihilation operators if the following conditions are satisfied:
(1) $(\mathbf{1}-B)(\mathbf{1}+\tilde{C})=0$ where $\tilde{c}_{i,j}^{k,l}=c_{j,l}^{i,k}$
(2) $(1+c_{i,i})(1-\tilde{b}_{i,i})=0$
(3) $c_{i,j}-\tilde{b}_{j,i}=0$ and $1-c_{i,j}\tilde{b}_{i,j}=0$ for $i\neq j$.

4. Fock representation

In this paper we want to introduce the scalar product, in which the creators are adjoint to annihilators. The last property is not satisfied for the usual scalar product on the $F[x_1,\ldots,x_n]$. So we have to introduce a modified inner product.

On the quotient algebra $\mathcal{A}$ the scalar product has to be degenerated, because the kernel of this product contains ideal J, every vector x which belongs to the ideal J in quotient algebra is to be $\bar{0}$ then $\langle x, x\rangle = 0$. It is well known that adjoint is an involution that gives a new relation between B, $\tilde{B}$ and tensor C is a Hermitian. For example, let us take:

$$d^+_{n+1,i}d^+_{n,j} - \sum_{k,l=1} b^{k,l}_{i,j} d^+_{n+1,k} d^+_{n,l} = 0$$

then acting by adjoint operation onto this relation we have:

$$d_{n,j}d_{n+1,i} - \sum_{k,l=1} \bar{b}^{k,l}_{i,j} d_{n,l} d_{n+1,k} = 0$$

from the other side we have:

$$d_{n,j}d_{n+1,i} - \sum_{k,l=1} \tilde{b}^{l,k}_{j,i} d_{n,l} d_{n+1,k} = 0$$

so we obtain $\bar{b}^{k,l}_{i,j} = \tilde{b}^{l,k}_{j,i}$.

In an analogous way we get:

$$c^{k,l}_{i,j} = \bar{c}^{l,k}_{j,i}.$$

Now let us introduce the Hermitian bilinear form on the $F[x_1, \ldots, x_n]$. In this definition we will introduce inner product on the direct products and by linearity we will extend onto the whole space $F[x_1, \ldots, x_n]$.

$$\langle\cdot,\cdot\rangle_C : F[x_1, \ldots, x_n] \times F[x_1, \ldots, x_n] \longmapsto \mathcal{C}$$

such that:

$$\langle x_{i_1} \ldots x_{i_r}, x_{j_1} \ldots x_{j_s}\rangle_C = \begin{cases} 0 & \text{for } r \neq s \\ \langle x_{i_2} \ldots x_{i_r}, a_{s,x_{i_1}} x_{j_1} \ldots x_{j_s}\rangle_C & \text{for } r = s. \end{cases}$$

According to [9] we can write:

$$\langle x, y\rangle_C = \langle x, Py\rangle_0$$

where

$$P_n = (\mathbf{1} \otimes P_{n-1})R_n,\ R_n = \mathbf{1} + C_1 + C_1C_2 + \cdots + C_1 \ldots C_n,\ P_0 = \mathbf{1}.$$

The following theorem given by Bożejko and Speicher [5] shows excellent criteria for positive defined Hermitian bilinear forms defined as below.

Theorem 4.1 (Bożejko–Speicher). If tensor $C \in \mathrm{Hom}(E \otimes E, E \otimes E)$ fulfils the following conditions:

(1) $C^{(i)}C^{(j)} = C^{(j)}C^{(i)}$ for $|i-j|\rangle 1$,
(2) $C^{(i)}C^{(j)}C^{(i)} = C^{(j)}C^{(i)}C^{(j)}$ for $|i-j| = 1$,
(3) $\|C\|\langle 1$,
(4) $c^{k,l}_{i,j} = \bar{c}^{l,k}_{j,i}$,
then $\langle\cdot,\cdot\rangle_C$ is strictly positive defined.

If $\|C\| = 1$ the above conditions are satisfied, then $\langle\cdot,\cdot\rangle_C$ is positive defined.

Directly from definition we see that creators are symmetric to annihilators

$$\begin{aligned}\langle a^+_{n,x} x_{i_1} \ldots x_{i_r}, x_{j_1} \ldots x_{j_s}\rangle_C &= \langle x x_{i_1} \ldots x_{i_r}, x_{i_1} \ldots x_{j_s}\rangle_C \\ &= \langle x_{i_1} \ldots x_{i_r}, a_{n,x} x_{j_1} \ldots x_{j_s}\rangle_C.\end{aligned}$$

Let

$$\ker\langle\cdot,\cdot\rangle_C = \{x \in F[x_1, \ldots, x_n] | \langle x, x\rangle_C = 0\}$$

then we have the following.

Proposition 4.2. $J \subset \ker\langle\cdot,\cdot\rangle_C \subset F[x_1, \ldots, x_n]$.

Proof. We prove this theorem by induction. For $m = 2$ we have

$$\begin{aligned}\overline{\langle(\mathbf{1}-B)x_{i_1}x_{i_2}, x_{j_1}x_{j_2}\rangle_C} &= \langle x_{j_1}x_{j_2}, (\mathbf{1}-B)x_{i_1}x_{i_2}\rangle_C \\ &= \langle x_{j_2}, a_{2,x_{j_1}}(\mathbf{1}-B)x_{i_1}x_{i_2}\rangle_C \\ &= \langle x_{j_2}, \bar{0}\rangle_C \\ &= \langle x_{j_2}, \bar{0}\rangle = 0\end{aligned}$$

so we have

$$\langle x, x\rangle_C = 0 \qquad \text{for } x \in I_2.$$

Let us take $\langle x_{i_1}\dots x_{i_m}, x_{j_1}\dots x_{j_m}\rangle_C = 0$ for $x_{i_1}\dots x_{i_m} \in I_m$, for certain $n \in \mathcal{N}$. We show that this relation is valid for $m+1$. Let $x_{i_1}\dots x_{i_{m+1}} \in I_{m+1}$ then:

$$\begin{aligned}\overline{\langle x_{i_1}\dots x_{i_{m+1}}, x_{j_1}\dots x_{j_{m+1}}\rangle_C} &= \langle x_{j_1}\dots x_{j_{m+1}}, x_{i_1}\dots x_{i_{m+1}}\rangle_C \\ &= \langle x_{j_2}\dots x_{j_{m+1}}, a_{m+1,x_{j_1}}x_{i_1}\dots x_{i_{m+1}}\rangle_C.\end{aligned}$$

We know that:

$$x_{i_1}\dots x_{i_{m+1}} \in I_{m+1} \Longrightarrow a_{n+1,x_1}x_{i_1}\dots x_{i_{n+1}} \in I_m$$

then by induction and linearity, for every $y \in F_{m+1}[x_1,\dots,x_n]$ we have:

$$\langle x_{i_1}\dots x_{i_{m+1}}, y\rangle_C = 0$$

and once again by applying linearity we have assertion, which completes the proof. □

Lemma 4.1. Let $x \in \ker\langle\cdot,\cdot\rangle_C$ then $a_i(x) \in \ker\langle\cdot,\cdot\rangle_C$ and $a_i^+(x) \in \ker\langle\cdot,\cdot\rangle_C$.

Proof. From assumption we know that space E is finitely dimensional. Let $\langle x, x\rangle_C = 0$, so from the Schwarz inequality we have:

$$\langle a_i(x), a_i(x)\rangle_C = \langle a_i^+a_i(x), x\rangle_C \leqslant \langle a_i^+a_i(x), a_i^+a_i(x)\rangle_C\langle x, x\rangle_C = 0.$$

The proof of the second part is analogous. □

Let us observe that $[a_i, a_j]_{\tilde{B}}x \in I \subset \ker\langle\cdot,\cdot\rangle_C$ and $[a_i, a_j^+]_Bx \in I \subset \ker\langle\cdot,\cdot\rangle_C$, $([a_i, a_j^+]_B - \delta_{i,j}\mathbf{1})x = 0 \in \ker\langle\cdot,\cdot\rangle_C$.

From lemma 4.1, the above observation follows that on the algebra $F[x_1,\dots,x_n]/\ker\langle\cdot,\cdot\rangle_C$ we can define $d_i, d_j^+ \in \mathrm{End}(F[x_1,\dots,x_n]/\ker\langle\cdot,\cdot\rangle_C)$, which satisfy:

(1) the following diagrams:

$$\begin{tikzcd} F_m[x_1,\dots,x_n] \arrow[r, mapsto, "a_{m,i}"] \arrow[d, "\pi_m"'] & F_{m-1}[x_1,\dots,x_n] \arrow[d, "\pi_{m-1}"] \\ \mathcal{A}_m \arrow[r, mapsto, "d_{m,i}"'] & \mathcal{A}_{m-1} \end{tikzcd} \qquad \begin{tikzcd} F_m[x_1,\dots,x_n] \arrow[r, mapsto, "a^+_{m,i}"] \arrow[d, "\pi_m"'] & F_{m+1}[x_1,\dots,x_n] \arrow[d, "\pi_{m+1}"] \\ \mathcal{A}_m \arrow[r, mapsto, "d^+_{m,i}"'] & \mathcal{A}_{m+1} \end{tikzcd}$$

(2) and commutation relations:

$$[d_i, d_j^+]_C = \delta_{i,j}\mathbf{1} \qquad [d_i, d_j]_{\tilde{B}} = 0 \qquad [d_i^+, d_j^+]_B = 0.$$

Theorem 4.3. If $\|C\| \leqslant 1$, where $C \in \mathrm{End}(F_2[x_1,\dots,x_n])$, then for every $f \in E$ the operators $\Phi(f) := (d(f) + d^+(f))/\sqrt{2}$ are essentially self-adjoint on the twisted Fock space.

Proof. To prove this, it is sufficient to find a dense set of analytic vectors for $\Phi(f)$ and every $f \in E \equiv F_1[x_1, \dots, x_n]$. Let us observe that for every $x \in \mathcal{A}_k$ we have:

$$\begin{aligned} \|d^+(f)x\|_C^2 &= \langle fx, fx\rangle_C \\ &= \langle fx, P_{k+1} fx\rangle_0 \\ &= \langle fx, (\mathbf{1}\otimes P_k)R_{k+1} fx\rangle_0 \\ &\leqslant \|R_{k+1}\|\langle fx, fP_k x\rangle_0 \\ &\leqslant (k+1)\|f\|^2\|x\|_C^2. \end{aligned}$$

Then we can write

$$\|d^\#(f)\dots d^\#(f)x\| \leqslant \sqrt{k+n}\dots\sqrt{k+1}\|f\|\|x\|_C$$

where $\# \in \{+, \}$ and

$$\left\|\sum_{n=0}^{\infty}\frac{t^n}{n!}\Psi^n(f)(x)\right\| \leqslant \sum_{n=0}^{\infty}\frac{t^n}{n!}2^{n/2}(k+n)^{n/2}\|f\|^n\|x\|_C$$

where $x \in \mathcal{A}_k$. Thus, the convergence radius is

$$r = \left(\sqrt{e}\|f\|\right)^{-1} > 0$$

which completes the proof. □

5. Examples

In this section we would like to show a few examples, which fulfil the consistency conditions formulated in section 3. We start with two simple examples.

Example 1 (bosons).

$$b_{i,j}^{k,l} = \tilde{b}_{i,j}^{k,l} = c_{i,j}^{k,l} = \delta_{i,l}\delta_{j,k}.$$

Example 2 (fermions).

$$b_{i,j}^{k,l} = \tilde{b}_{i,j}^{k,l} = c_{i,j}^{k,l} = -\delta_{i,l}\delta_{j,k}.$$

In the general case a linear condition is satisfied when $\mathrm{Im}(id_{\mathcal{E}\otimes\mathcal{E}} - B) \subset \mathrm{Ker}(id_{\mathcal{E}\otimes\mathcal{E}} + C)$. For example this relation is not satisfied when:

$$B(x_i x_j) = q^{i-j}x_j x_i \qquad \tilde{B}(x_i x_j) = q^{i-j}x_j x_i \qquad C(x_i x_j) = q^{i-j}x_j x_i.$$

This relation satisfies the second condition of theorem 3.3. For $q \in \{-1, 1\}$ all the assumptions of theorems 3.3 and 3.4 are satisfied. Then we have the following.

Example 3 (mixed bosons–fermions).

$$b_{i,j}^{k,l} = \tilde{b}_{i,j}^{k,l} = c_{i,j}^{k,l} = (-1)^{i-j}\delta_{i,l}\delta_{j,k}$$

and, more generally we get the following

Example 4 (q-deformed algebra).

$$b_{i,j}^{k,l} = \tilde{b}_{i,j}^{k,l} = q^{j-i}\delta_{i,l}\delta_{j,k} \qquad c_{i,j}^{k,l} = q^{i-j}\delta_{i,l}\delta_{j,k} \qquad \text{for } q \in \mathcal{R}$$

as a particular case of more general relations, which were studied in [10].

Example 5 (bosonic colour statistics). Let $|q_i| = 1$ for $i \in \{1 \dots \dim E\}$.

$$b_{i,j}^{k,l} = \tilde{b}_{i,j}^{k,l} = q_j\bar{q}_i\delta_{i,l}\delta_{j,k} \qquad c_{i,j}^{k,l} = q_i\bar{q}_j\delta_{i,l}\delta_{j,k}.$$

Example 6 (fermionic colour statistics). Let $|q_i| = 1$ for $i \in \{1 \ldots \dim E\}$.

$$b^{k,l}_{i,j} = \tilde{b}^{k,l}_{i,j} = -q_j \bar{q}_i \delta_{i,l} \delta_{j,k} \qquad c^{k,l}_{i,j} = -q_i \bar{q}_j \delta_{i,l} \delta_{j,k}.$$

The following example was given by Farlie and Nuyts [7].

Example 7. Let

$$A_r = \{q \in \mathcal{C} : (q - q^{-1})\overline{(q - q^{-1})} = r^2 \in R_+\}$$

then

$$\tilde{b}_{i,j} = b = \frac{q_i - q_i^{-1}}{q_j - q^{-1}} = c_{j,i} \qquad \text{for } q_i, q_j \in A_r.$$

We can give a slightly more general example.

Example 8. Let $Q \in Mat_n(\mathcal{C})$ such that $q_{i,j}\tilde{q}_{i,j} = 1$ and $q_{i,j} = \tilde{q}_{j,i}$ then

$$\tilde{b}_{i,j} = b_{i,j} = q_{i,j} \qquad c_{i,j} = q_{j,i}.$$

The following example was given by Marcinek [27].

Example 9 (quons). Let $Q \in Mat_n(\mathcal{C})$ such that $q_{i,j}\tilde{q}_{i,j} = 1$ and $q_{i,j} = \tilde{q}_{j,i}$ then

$$\begin{aligned} &\tilde{b}_{i,j} = b_{i,j} = q_{i,j} c_{i,j} = q_{j,i} && \text{for } i \neq j \\ &\tilde{b}_{i,i} = b_{i,i} = 1, c_{i,i} = q_i && \text{where } q_i \in \mathcal{C}. \end{aligned}$$

Example 10 ($SU_q(n)$ algebra). Let

$$\tilde{b}_{i,j} = b_{i,j} = 1 \qquad c_{i,j} = q^{2\delta_{i,j}} \qquad i, j \in \{1, \ldots n\}$$

then by applying corollary 3.2 we have the operators b_i, b_i^+ with the following relations:

$$b_i b_j^+ - q^{2\delta_{i,j}} b_j^+ b_i = \delta_{i,j} \mathbf{1} \qquad b_i b_j - b_j b_i = 0 = b_i^+ b_j^+ - b_j^+ b_i^+.$$

Thus, following Chaichian *et al* (see [12, 13]), we define the operators:

$$A_i = q^{\sum_{k>i} N_k} b_i \qquad A_i^+ = b_i q^{\sum_{k>i} N_k}$$

where N_i is defined by $|K_i| = q^{N_i + \omega_i}$ and $K_i = b_i b_i^+ - b_i^+ b_i$, here b_i, b_i^+ form the Fock representation and $K_i > 0$, this representation is bounded for $0 < q < 1$. We obtain $SU_q(n)$-covariant algebra studied by Woronowicz and Pusz [16] and Chaichian *et al* [17]:

$$\begin{aligned} &A_i A_j = q A_j A_i \qquad A_j^+ A_i^+ = q A_i^+ A_j^+ \qquad i < j \\ &A_i A_j^+ = q A_j^+ A_i \qquad i \neq j \\ &A_i A_i^+ - q^2 A_i^+ A_i = 1 - (1 - q^2) \sum_{k>i} A_k^+ A_k. \end{aligned}$$

Acknowledgments

The author would like to thank M Bożejko, T Lulek, R Lenczewski and W Marcinek for discussions and critical remarks. Support from Polish National KBN grant 2P03B12211 is gratefully acknowledged.

References

[1] Greenberg O W 1990 *Phys. Rev. Lett.* **64** 705

[2] Greenberg O W 1991 Particles with small violations of Fermi or Bose statistics *Phys. Rev.* D **43** 4111

[3] Mohopatra R N 1990 *Phys. Lett.* B **242** 407

[4] Bożejko M and Speicher R 1991 An example of a generalized Brownian motion *Comm. Math. Phys.* **137** 519–31

[5] Bożejko M and Speicher R 1994 Completely positive maps on Coxeter groups, deformed commutation relations and operator spaces *Math. Ann.* **300** 97–120

[6] Fivel D 1990 Interpolation between Fermi and Bose statistics using generalised commutators *Phys. Rev. Lett.* **65** 3361–4

[7] Farlie D and Nuyts J 1993 *J. Math. Phys.* **34** 4441

[8] Lenczewski R and Podgórski K 1992 A q-analog of the quantum central limit theorem for $SU_q(2)$ *J. Math. Phys.* **33** 2768–78

[9] Jörgensen P E T, Schmith L M and Werner R F 1995 *J. Funct. Anal.* **134**

[10] Marcinek W 1994 *Rep. Math. Phys.* **34** 325–40

[11] Borowiec A and Kharchenko V K 1995 Coordinate calculi on associative algebras *Quantum Groups* ed J Lukierski *et al* (Walsaw: Polish Scientific Publishers) pp 231–42

[12] Chaichian M, Grosse H and Presnajder P 1994 Unitary representations of the q-oscillator algebra *J. Phys. A: Math. Gen.* **27** 2045

[13] Chaichian M, Grosse H and Presnajder P 1995 Representations and some applications of q-oscillators *Quantum Groups* ed J Lukierski *et al* (Warsaw: Polish Scientific Publishers) pp 183–91

[14] Borowiec A, Kharchenko V K and Oziewicz Z 1994 On free differentials on associative algebras *Non-Associative Algebra and its Applications (Mathematics and its Applications vol 303)* ed S Gonzales (Dordrecht: Kluwer) pp 46–53

[15] Wess J and Zumino B 1990 Covariant differential calculus on the quantum hyperplane *Nucl. Phys. (Proc. Suppl.)* B **18** 302–12

[16] Woronowicz S and Pusz W 1989 Twisted second quantization *Rep. Math. Phys.* **27** 231–57

[17] Chaichian M, Kulish P and Lukierski J 1991 Supercovariant systems of q-oscillators and Q-symmetric hamiltonians *Phys. Lett.* B **262** 43

[18] Woronowicz S 1989 Differential calculus on compact matrix pseudogroups (quantum groups) *Commun. Math. Phys.* **122** 125–70

[19] Pusz W 1989 *Rep. Math. Phys.* **27** 394

[20] Connes A 1985 Non-commutative differential geometry *Publ. Math. IHES* **62** 257–360

[21] Wilczek F 1982 *Phys. Rev. Lett.* **48** 114

[22] Wilczek F and Zee A 1983 *Phys. Rev. Lett.* **51** 2250

[23] Wu Y S 1984 *Phys. Rev. Lett.* **52** 2103

[24] Marcinek W and Rałowski R 1995 Particle operators from braided geometry *Quantum Groups* ed J Lukierski *et al* (Warsaw: Polish Scientific Publishers) pp 149–53

[25] Marcinek W and Rałowski R 1995 On Wick algebra's with braid relations *J. Math. Phys.* **36** 2803–12

[26] Rałowski R 1995 On deformations of commutation relation algebras *Preprint* ITP UWr no 890

[27] Marcinek W On commutation relation for quons *Preprint* q-alg 9512015 and submitted to *Rep. Math. Phys.*

[28] Marcinek W 1997 On quantum Weyl algebras and generalised quons *Proc. Symp. Quantum Groups and Quantum Spaces (Warsaw, November 20–29, 1995)* ed R Budzynski, W Pusz and S Zakrzewski (Warsaw: Banach)

[29] Oziewicz Z and Sitarczyk Cz 1992 Parallel treatment of Riemannian and symplectic Clifford algebra's *Clifford Algebra's and their Applications in Mathematical Physics* ed A Micali *et al* (Deventer: Kluwer) pp 83–95